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A generalized auxiliary equation method and its application to (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations

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Abstract
A generalized auxiliary equation method is proposed to construct more general exact solutions of nonlinear partial differential equations. With the aid of symbolic computation, we choose the (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations to illustrate the validity and advantages of the method. As a result, many new and more general exact non-travelling wave and coefficient function solutions are obtained including soliton-like solutions, triangular-like solitions, single and combined non-degenerate Jacobi elliptic doubly-like periodic solutions, and Weierstrass elliptic doubly-like periodic solutions.

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1. Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. With the development of soliton theory, various methods for obtaining exact solutions of NLPDEs have been presented, such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], tanh function method [5, 6], sine–cosine method [7], homogenous balance method [8], homotopy perturbation method [9], variational method [10], asymptotic methods [11], non-perturbative methods [12], Exp-function method [13], Adomian Pade approximation [14], Jacobi elliptic function expansion method [15], F-expansion method [16, 17], Weierstrass semi-rational expansion method [18], unified rational expansion method [19], algebraic method [20–23], auxiliary equation method [24–27] and so on. Recently, Sirendaoreji [28] and Huang *et al* [29], respectively, proposed a new auxiliary equation method by introducing a new first-order nonlinear ordinary differential equation with six-degree nonlinear term and its solutions to construct exact travelling wave solutions of NLPDEs in a unified way.

The present paper is motivated by the desire to generalize the work done in [20–29] to construct new and more general exact solutions which contain not only the results obtained by using the methods in [20–29] but also a series of new and more general exact solutions, in which the restriction on ξ as merely a linear function and the restriction on coefficients being constants are removed. For illustration, we apply this method to the (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations and successfully obtain many new and more general exact solutions.

The rest of this paper is organized as follows: in section 2, we give the description of the generalized auxiliary equation method; in section 3, we apply this method to the (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations; in section 4, some conclusions are given.

2. A generalized auxiliary equation method

In this section, we outline a generalized auxiliary equation method. For a given NLPDE with independent variables $x = (t, x_1, x_2, \ldots, x_m)$ and dependent variable u:

$$F\big(u, u_t, u_{x_1}, u_{x_2}, \ldots, u_{x_m}, u_{x_1t}, u_{x_2t} \ldots, u_{x_mt}, u_{tt}, u_{x_1x_1}, u_{x_2x_2}, \ldots, u_{x_mx_m}, \ldots\big) = 0, \tag{1}$$

we seek its solutions in the more general form:

$$u = a_0 + \sum_{i=1}^{n}\{a_i\phi^{-i}(\xi) + b_i\phi^i(\xi) + c_i\phi^{i-1}(\xi)\phi'(\xi) + d_i\phi^{-i}(\xi)\phi'(\xi)\}, \tag{2}$$

with $\phi(\xi)$ satisfying the new auxiliary equation:

$$\phi'^2(\xi) = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi) + h_5\phi^5(\xi) + h_6\phi^6(\xi), \tag{3}$$

where $a_0 = a_0(x), a_i = a_i(x), b_i = b_i(x), c_i = c_i(x), d_i = d_i(x) (i = 1, 2, \ldots, n)$ and $\xi = \xi(x)$ are functions to be determined, h_j $(j = 0, 1, 2 \ldots, 6)$ are real constants. To determine u explicitly, we take the following four steps.

Step 1. Determine the integer n. Substituting (2) along with (3) into equation (1) and balancing the highest order partial derivative with the nonlinear terms in equation (1), we then obtain the value of n. For example, in the case of KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{4}$$

we have $n = 4$.

Step 2. Derive a system of equations. Substituting (2) given the value of n obtained in step 1 along with (3) into equation (1), collecting coefficients of $\phi^j(\xi)\phi'^l(\xi)$ $(l = 0, 1;\ j = 0, \pm1, \pm2, \ldots)$, then setting each coefficient to zero, we can derive a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i and ξ.

Step 3. Solve the system of equations. Solving the system of over-determined partial differential equations obtained in step 2 by use of *Mathematica*, we can obtain the explicit expressions for a_0, a_i, b_i, c_i, d_i and ξ.

Step 4. Obtain exact solutions. By using the results obtained in the above steps, we can derive a series of fundamental solutions of equation (1) depending on the solution $\phi(\xi)$ of equation (3).

By choosing the different values of h_j $(j = 0, 1, 2 \ldots, 6)$, equation (3) has many kinds of special solutions. Some of them are listed in [22] under the condition $h_5 = h_6 = 0$. In order to find the solutions with $h_6 \neq 0$ of equation (3) conveniently, we set

$$\phi(\xi) = \varphi^{1/2}(\xi), \tag{5}$$

then equation (3) becomes

$$\varphi'^2(\xi) = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 = 4(h_0\varphi(\xi) + h_1\varphi^{3/2}(\xi) + h_2\varphi^2(\xi) + h_3\varphi^{5/2}(\xi) + h_4\varphi^3(\xi) + h_5\varphi^{7/2}(\xi) + h_6\varphi^4(\xi)). \tag{6}$$

With the aid of equations (5) and (6), we can easily find some special solutions with $h_6 \neq 0$ of equation (3), which are listed as follows.

Case I. Suppose that $h_1 = h_3 = h_5 = 0$, $h_0 = \frac{8h_2^2}{27h_4}$ and $h_6 = \frac{h_4^2}{4h_2}$.

(i) If $h_2 < 0$ and $h_4 > 0$, then equation (3) has the following solutions (here and thereafter $\varepsilon = \pm 1$):

$$\phi(\xi) = \left\{-\frac{8h_2 \tanh^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))}{3h_4[3 + \tanh^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))]}\right\}^{1/2}, \tag{7}$$

$$\phi(\xi) = \left\{-\frac{8h_2 \coth^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))}{3h_4[3 + \coth^2(\varepsilon\sqrt{-h_2/3}(\xi + \xi_0))]}\right\}^{1/2}. \tag{8}$$

(ii) If $h_2 > 0$ and $h_4 < 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{\frac{8h_2 \tan^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))}{3h_4[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))]}\right\}^{1/2}, \tag{9}$$

$$\phi(\xi) = \left\{\frac{8h_2 \cot^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))}{3h_4[3 - \cot^2(\varepsilon\sqrt{h_2/3}(\xi + \xi_0))]}\right\}^{1/2}. \tag{10}$$

Case II. Suppose that $h_0 = h_1 = h_3 = h_5 = 0$ and $h_6 \neq 0$.

(i) If $h_2 > 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{-\frac{h_2h_4 \,\mathrm{sech}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4^2 - h_2h_6[1 + \varepsilon \tanh(\sqrt{h_2}(\xi + \xi_0))]^2}\right\}^{1/2}, \tag{11}$$

$$\phi(\xi) = \left\{\frac{h_2h_4 \,\mathrm{csch}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4^2 - h_2h_6[1 + \varepsilon \coth(\sqrt{h_2}(\xi + \xi_0))]^2}\right\}^{1/2}. \tag{12}$$

(ii) If $h_2 > 0$ and $h_6 > 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{-\frac{h_2 \,\mathrm{sech}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{h_2h_6}\tanh(\sqrt{h_2}(\xi + \xi_0))}\right\}^{1/2}, \tag{13}$$

$$\phi(\xi) = \left\{\frac{h_2 \,\mathrm{csch}^2(\sqrt{h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{h_2h_6}\coth(\sqrt{h_2}(\xi + \xi_0))}\right\}^{1/2}. \tag{14}$$

(iii) If $h_2 < 0$ and $h_6 > 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{ -\frac{h_2 \sec^2(\sqrt{-h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{-h_2 h_6}\tan(\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{15}$$

$$\phi(\xi) = \left\{ -\frac{h_2 \csc^2(\sqrt{-h_2}(\xi + \xi_0))}{h_4 + 2\varepsilon\sqrt{-h_2 h_6}\cot(\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{16}$$

Case III. Suppose that $h_0 = h_1 = h_3 = h_5 = 0$, $h_6 \neq 0$ and $h_4^2 - 4h_2h_6 > 0$.

(i) If $h_2 > 0$, then equation (3) has the following solution:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{sech}(2\sqrt{h_2}(\xi + \xi_0))}{\varepsilon\sqrt{h_4^2 - 4h_2h_6} - h_4 \operatorname{sech}(2\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{17}$$

(ii) If $h_2 < 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{2h_2 \sec(2\sqrt{-h_2}(\xi + \xi_0))}{\varepsilon\sqrt{h_4^2 - 4h_2h_6} - h_4 \sec(2\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{18}$$

$$\phi(\xi) = \left\{ \frac{2h_2 \csc(2\sqrt{-h_2}(\xi + \xi_0))}{\varepsilon\sqrt{h_4^2 - 4h_2h_6} - h_4 \csc(2\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{19}$$

(iii) If $h_2 > 0$, $h_4 < 0$ and $h_6 < 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{sech}^2(\varepsilon\sqrt{h_2}(\xi + \xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right)\operatorname{sech}^2(\varepsilon\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{20}$$

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{csch}^2(\varepsilon\sqrt{h_2}(\xi + \xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} + \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right)\operatorname{csch}^2(\varepsilon\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{21}$$

(iv) If $h_2 < 0$, $h_4 > 0$ and $h_6 < 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{ \frac{-2h_2 \sec^2(\varepsilon\sqrt{-h_2}(\xi + \xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} - h_4\right)\sec^2(\varepsilon\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}, \tag{22}$$

$$\phi(\xi) = \left\{ \frac{2h_2 \csc^2(\varepsilon\sqrt{-h_2}(\xi + \xi_0))}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right)\csc^2(\varepsilon\sqrt{-h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{23}$$

Case IV. Suppose that $h_0 = h_1 = h_3 = h_5 = 0$, $h_6 \neq 0$ and $h_4^2 - 4h_2h_6 < 0$.

(1) If $h_2 > 0$, then equation (3) has the following solution:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{csch}(2\sqrt{h_2}(\xi + \xi_0))}{\varepsilon\sqrt{4h_2h_6 - h_4^2} - h_4 \operatorname{csch}(2\sqrt{h_2}(\xi + \xi_0))} \right\}^{1/2}. \tag{24}$$

Case V. Suppose that $h_0 = h_1 = h_3 = h_5 = 0$, $h_6 \neq 0$ and $h_4^2 - 4h_2h_6 = 0$.

(i) If $h_2 > 0$, then equation (3) has the following solutions:

$$\phi(\xi) = \left\{ -\frac{h_2}{h_4}[1 + \varepsilon \tanh(\varepsilon\sqrt{h_2}(\xi + \xi_0))] \right\}^{1/2}, \tag{25}$$

$$\phi(\xi) = \left\{ -\frac{h_2}{h_4}[1 + \varepsilon \coth(\varepsilon\sqrt{h_2}(\xi + \xi_0))] \right\}^{1/2}. \tag{26}$$

3. Application of the method

In this section, we would like to use our method to obtain new and more general exact solutions of the (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations:

$$u_t - u_{xxx} - 3(uv)_x = 0, \tag{27}$$

$$u_x - v_y = 0. \tag{28}$$

By using a generalized algebraic method, Chen *et al* obtained some soliton-like solutions and triangular-like solitions of equations (27) and (28) in [21].

According to step 1, we get $n = 4$ for u and v. We assume that equations (27) and (28) have the following formal solutions.

$$\begin{aligned} u &= a_0 + a_1\phi^{-1}(\xi) + a_2\phi^{-2}(\xi) + a_3\phi^{-3}(\xi) + a_4\phi^{-4}(\xi) + b_1\phi(\xi) + b_2\phi^2(\xi) + b_3\phi^3(\xi) \\ &\quad + b_4\phi^4(\xi) + c_1\phi'(\xi) + c_2\phi(\xi)\phi'(\xi) + c_3\phi^2(\xi)\phi'(\xi) + c_4\phi^3(\xi)\phi'(\xi) + d_1\phi^{-1}(\xi)\phi'(\xi) \\ &\quad + d_2\phi^{-2}(\xi)\phi'(\xi) + d_3\phi^{-3}(\xi)\phi'(\xi) + d_4\phi^{-4}(\xi)\phi'(\xi), \end{aligned} \tag{29}$$

$$\begin{aligned} v &= A_0 + A_1\phi^{-1}(\xi) + A_2\phi^{-2}(\xi) + A_3\phi^{-3}(\xi) + A_4\phi^{-4}(\xi) + B_1\phi(\xi) + B_2\phi^2(\xi) + B_3\phi^3(\xi) \\ &\quad + B_4\phi^4(\xi) + C_1\phi'(\xi) + C_2\phi(\xi)\phi'(\xi) + C_3\phi^2(\xi)\phi'(\xi) + C_4\phi^3(\xi)\phi'(\xi) + D_1\phi^{-1}(\xi)\phi'(\xi) \\ &\quad + D_2\phi^{-2}(\xi)\phi'(\xi) + D_3\phi^{-3}(\xi)\phi'(\xi) + D_4\phi^{-4}(\xi)\phi'(\xi), \end{aligned} \tag{30}$$

where $a_0 = a_0(y,t)$, $a_i = a_i(y,t)$, $b_i = b_i(y,t)$, $c_i = c_i(y,t)$, $d_i = d_i(y,t)$, $A_0 = A_0(y,t)$, $A_i = A_i(y,t)$, $B_i = B_i(y,t)$, $C_i = C_i(y,t)$, $D_i = D_i(y,t)$ $(i = 1,2,3,4)$, $\xi = \rho\omega + \eta$, $\rho = \rho(x)$, $\omega = \omega(y,t)$, $\eta = \eta(y,t)$.

With the aid of *Mathematica*, substituting (29) and (30) along with (3) into equations (27) and (28), then setting each coefficient of $\phi^j(\xi)\phi'^l(\xi)$ $(l = 0, 1;\ j = \pm 1, \pm 2, \ldots)$ to zero, we get a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i, A_0, A_i, B_i, C_i, D_i, ρ, ω and η as follows:

$$-30h_6\omega\rho'(b_4C_4 + B_4c_4 + 16h_6c_4\omega^2\rho'^2) = 0,$$

$$24h_0\omega\rho'(a_4D_4 + A_4d_4) = 0, \qquad d_{1,t} = 0, \qquad D_{1,y} = 0,$$

$$\begin{aligned} &15\omega\rho'[a_2A_3 + A_2a_3 + a_1A_4 + A_1a_4 + h_0(d_1D_4 + D_1d_4 + d_2D_3 + D_2d_3) \\ &\qquad + h_1(d_2D_4 + D_2d_4 + d_3D_3) + h_2(d_3D_4 + D_3d_4)] \\ &\qquad + 30\omega^3\rho'^3(2h_0a_3 + 3h_1a_4) - 60h_0d_4^2\omega^2\rho'\rho'' = 0, \end{aligned}$$

$$\begin{aligned} &-\tfrac{1}{2}\omega\rho'(h_1d_1 + 2h_2d_2 + 3h_3d_3 + 4h_4d_4) \\ &\qquad + \tfrac{1}{2}(\rho\omega_y + \eta_y)(h_1D_1 + 2h_2D_2 + 3h_3D_3 + 4h_4D_4) - A_{1,y} = 0, \end{aligned}$$

$$\begin{aligned} &-24h_6\omega\rho'(b_2C_4 + B_2c_4 + b_4C_2 + B_4c_2) - \tfrac{57}{2}h_5\omega\rho'(b_4C_4 + B_4c_4) \\ &\qquad + \tfrac{15}{2}h_5\omega^3\rho'^3(42h_6c_3 + 107h_5c_4) = 0, \end{aligned}$$

$$-24h_6\omega\rho'(b_2C_4+B_2c_4+b_3C_3+B_3c_3+b_4C_2+B_4c_2)$$
$$-\tfrac{51}{2}h_5\omega\rho'(b_3C_4+B_3c_4+b_4C_3+B_4c_3)$$
$$-27h_4\omega\rho'(b_4C_4+B_4c_4)+\tfrac{1}{4}\omega^3\rho'^3(768h_6^2c_2+2054h_5h_6c_3$$
$$+1309h_5^2c_4+2688h_4h_6c_4)=0,$$
$$-\tfrac{1}{2}\omega\rho'(10h_6c_3+11h_5c_4)-\tfrac{1}{2}(\rho\omega_y+\eta_y)(10h_6C_3+11h_5C_4)=0,$$
$$4b_4\omega\rho'-C_{4,y}-4B_4\rho\omega_y-4B_4\eta_y=0,\qquad -33h_6c_4C_4\omega\rho'=0,$$
$$6h_6(c_4\omega\rho'-C_4\rho\omega_y-C_4\eta_y)=0,$$
$$-4(a_4\omega\rho'-A_4\rho\omega_y-A_4\eta_y)=0,$$
$$\vdots$$

there are totally 66 equations in the set of over-determined partial differential equations, just some simple and central equations are shown here for convenience. Solving the system of over-determined partial differential equations by use of *Mathematica*, we obtain the following results.

Case 1.

$$a_0=\frac{\left(-k_1^2k_3^2h_2\pm6k_1^2k_3^2\sqrt{h_0h_4}-3k_4\right)f_1(y)}{3k_1k_3},\qquad a_1=-\frac{k_1k_3h_1f_1(y)}{2},$$
$$a_2=-k_1k_3h_0f_1(y), \tag{31}$$

$$a_3=0,\qquad a_4=0,\qquad b_1=-\frac{k_1k_3h_3f_1(y)}{2},\qquad b_2=-k_1k_3h_4f_1(y),$$
$$b_3=0,\qquad b_4=0, \tag{32}$$

$$c_1=\pm k_1k_3\sqrt{h_4}f_1(y),\qquad c_2=0,\qquad c_3=0,\qquad c_4=0,\qquad d_1=0,$$
$$d_2=\pm k_1k_3\sqrt{h_0}f_1(y), \tag{33}$$

$$d_3=0,\qquad d_4=0,\qquad A_0=\frac{3k_1k_3k_4+f_2'(t)}{3k_1k_3},\qquad A_1=-\frac{k_1^2k_3^2h_1}{2},$$
$$A_2=-k_1^2k_3^2h_0,\qquad A_3=0, \tag{34}$$

$$A_4=0,\qquad B_1=-\frac{k_1^2k_3^2h_3}{2},\qquad B_2=-k_1^2k_3^2h_4,\qquad B_3=0,$$
$$B_4=0,\qquad C_1=\pm k_1^2k_3^2\sqrt{h_4}, \tag{35}$$

$$C_2=0,\qquad C_3=0,\qquad C_4=0,\qquad D_1=0,\qquad D_2=\pm k_1^2k_3^2\sqrt{h_0},$$
$$D_3=0,\qquad D_4=0, \tag{36}$$

$$\rho=k_1x+k_2,\qquad \omega=k_3,\qquad \eta=\int f_1(y)\,\mathrm{d}y+f_2(t),\qquad h_5=0,$$
$$h_6=0,\qquad \pm h_3\sqrt{h_0}-h_1\sqrt{h_4}=0, \tag{37}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t)=\mathrm{d}f_2(t)/\mathrm{d}t$, k_1 and k_3 are nonzero constants, k_2 and k_4 are arbitrary constants. The sign '±' in C_1 and D_2 means that all possible combinations of '+' and '−' can be taken. If the same sign is used in C_1 and D_2, then '+' must be used in a_0 and '−' must be used in (37). If different signs are used in C_1 and D_2, then '−' must be used in a_0 and '+' must be used in (37). Furthermore, the same sign must be used in c_1 and C_1. Also the same sign must be use in d_2 and D_2. Hereafter, the sign '±' always stands for this meaning in the similar circumstances.

Case 2.

$$a_0 = -\frac{\left(3k_4 + k_1^2 k_3^2 h_2\right) f_1(y)}{3k_1 k_3}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = 0, \qquad b_1 = -\frac{k_1 k_3 h_3 f_1(y)}{2}, \tag{38}$$

$$b_2 = -k_1 k_3 h_4 f_1(y), \qquad b_3 = 0, \qquad b_4 = 0, \qquad c_1 = \pm k_1 k_3 \sqrt{h_4} f_1(y), \qquad c_2 = 0, \qquad c_3 = 0, \tag{39}$$

$$c_4 = 0, \qquad d_1 = 0, \qquad d_2 = 0, \qquad d_3 = 0, \qquad d_4 = 0, \qquad A_0 = \frac{3k_1 k_3 k_4 + f_2'(t)}{3k_1 k_3}, \qquad A_1 = 0, \qquad A_2 = 0, \tag{40}$$

$$A_3 = 0, \qquad A_4 = 0, \qquad B_1 = -\frac{k_1^2 k_3^2 h_3}{2}, \qquad B_2 = -k_1^2 k_3^2 h_4, \qquad B_3 = 0, \qquad B_4 = 0, \qquad C_1 = \pm k_1^2 k_3^2 \sqrt{h_4}, \tag{41}$$

$$C_2 = 0, \qquad C_3 = 0, \qquad C_4 = 0, \qquad D_1 = 0, \qquad D_2 = 0, \qquad D_3 = 0, \qquad D_4 = 0, \tag{42}$$

$$\rho = k_1 x + k_2, \qquad \omega = k_3, \qquad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \qquad h_5 = 0, \qquad h_6 = 0, \tag{43}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k_1 and k_3 are nonzero constants, k_2 and k_4 are arbitrary constants.

Case 3.

$$a_0 = -\frac{\left(4k_1^2 k_3^2 h_2 + 3k_4\right) f_1(y)}{3k_1 k_3}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = 0, \qquad b_1 = 0, \tag{44}$$

$$b_2 = -2k_1 k_3 h_4 f_1(y), \qquad b_3 = 0, \qquad b_4 = -4k_1 k_3 h_6 f_1(y), \qquad c_1 = 0, \qquad c_2 = \pm 4k_1 k_3 \sqrt{h_6} f_1(y), \tag{45}$$

$$c_3 = 0, \qquad c_4 = 0, \qquad d_1 = 0, \qquad d_2 = 0, \qquad d_3 = 0, \qquad d_4 = 0, \qquad A_0 = \frac{3k_1 k_3 k_4 + f_2'(t)}{3k_1 k_3}, \qquad A_1 = 0, \tag{46}$$

$$A_2 = 0, \qquad A_3 = 0, \qquad A_4 = 0, \qquad B_1 = 0, \qquad B_2 = -2k_1^2 k_3^2 h_4, \qquad B_3 = 0, \qquad B_4 = -4k_1^2 k_3^2 h_6, \tag{47}$$

$$C_1 = 0, \qquad C_2 = \pm 4k_1^2 k_3^2 \sqrt{h_6}, \qquad C_3 = 0, \qquad C_4 = 0, \qquad D_1 = 0, \qquad D_2 = 0, \qquad D_3 = 0, \qquad D_4 = 0, \tag{48}$$

$$\rho = k_1 x + k_2, \qquad \omega = k_3, \qquad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \qquad h_0 = h_0,$$
$$h_1 = 0, \qquad h_3 = 0, \qquad h_5 = 0, \tag{49}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k_1 and k_3 are nonzero constants, k_2 and k_4 are arbitrary constants.

Case 4.

$$a_0 = -\frac{\left(4k_1^2k_3^2h_2 + 3k_4\right)f_1(y)}{3k_1k_3}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0,$$
$$a_4 = 0, \qquad b_1 = 0, \tag{50}$$

$$b_2 = -4k_1k_3h_4 f_1(y), \qquad b_3 = 0, \qquad b_4 = -8k_1k_3h_6 f_1(y), \qquad c_1 = 0,$$
$$c_2 = 0, \qquad c_3 = 0, \tag{51}$$

$$c_4 = 0, \qquad d_1 = 0, \qquad d_2 = 0, \qquad d_3 = 0, \qquad d_4 = 0,$$
$$A_0 = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3}, \qquad A_1 = 0, \tag{52}$$

$$A_2 = 0, \qquad A_3 = 0, \qquad A_4 = 0, \qquad B_1 = 0, \qquad B_2 = -4k_1^2k_3^2h_4, \qquad B_3 = 0,$$
$$B_4 = -8k_1^2k_3^2h_6, \tag{53}$$

$$C_1 = 0, \qquad C_2 = 0, \qquad C_3 = 0, \qquad C_4 = 0, \qquad D_1 = 0, \qquad D_2 = 0,$$
$$D_3 = 0, \qquad D_4 = 0, \qquad \rho = k_1x + k_2, \tag{54}$$

$$\omega = k_3, \qquad \eta = \int f_1(y)\,\mathrm{d}y + f_2(t), \qquad h_4^2 = 4h_2h_6, \qquad h_0 = 0,$$
$$h_1 = 0, \qquad h_3 = 0, \qquad h_5 = 0, \tag{55}$$

where $f_1(y)$ and $f_2(t)$ are arbitrary functions of y and t respectively, $f_2'(t) = \mathrm{d}f_2(t)/\mathrm{d}t$, k_1 and k_3 are nonzero constants, k_2 and k_4 are arbitrary constants.

From (29) and (30), cases 1–2 and cases I–V in [22], we can obtain many kinds of solutions of equations (27) and (28) depending on the special choice for $h_i (i = 0, 1, 2, \ldots, 6)$.

3.1. If $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 2rq + p^2$, $h_3 = 2pq$, $h_4 = q^2$, $h_5 = h_6 = 0$, then $\phi(\xi)$ is one of the 24 $\phi_l^{\mathrm{I}}(l = 1, 2, \ldots, 24)$

For example, if we select $l = 10$, from case 1 we obtain soliton-like solutions of equations (27) and (28):

$$\begin{aligned} u = {} & \frac{\left[k_1^2k_3^2(-2qr - p^2 \pm 6|qr|) - 3k_4\right]f_1(y)}{3k_1k_3} - \frac{1}{2}k_1k_3pf_1(y)\,\mathrm{sech}(M\xi) \\ & \times [M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M] - \frac{1}{4}k_1k_3f_1(y)\,\mathrm{sech}^2(M\xi) \\ & \times [M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]^2 - \frac{2k_1k_3pqrf_1(y)\cosh(M\xi)}{[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]} \end{aligned}$$

$$- \frac{4k_1k_3q^2r^2 f_1(y)\cosh^2(M\xi)}{[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]^2} \pm \frac{2k_1k_3|q|rM^2 f_1(y)[-1 \pm \mathrm{i}\sinh(M\xi)]}{[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]^2}$$

$$\pm \frac{1}{2}k_1k_3\varepsilon M^2 f_1(y)\,\mathrm{sech}^2(M\xi)[-1 \pm \mathrm{i}\sinh(M\xi)],$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{1}{2}k_1^2k_3^2 p\,\mathrm{sech}(M\xi)[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]$$

$$- \frac{1}{4}k_1^2k_3^2\,\mathrm{sech}^2(M\xi)[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]^2$$

$$- \frac{2k_1^2k_3^2 pqr\cosh(M\xi)}{[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]} - \frac{4k_1^2k_3^2q^2r^2\cosh^2(M\xi)}{[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]^2}$$

$$\pm \frac{2k_1^2k_3^2|q|rM^2[-1 \pm \mathrm{i}\sinh(M\xi)]}{[M\sinh(M\xi) - p\cosh(M\xi) \pm \mathrm{i}M]^2} \pm \frac{1}{2}k_1^2k_3^2\varepsilon M^2\,\mathrm{sech}^2(M\xi)[-1 \pm \mathrm{i}\sinh(M\xi)],$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,\mathrm{d}y + f_2(t)$, $M = \sqrt{p^2 - 4qr}$. If '+' is used in a_0, then $qr < 0$. If '−' is used in a_0, then $qr > 0$.

3.2. If $h_0 = r^2$, $h_1 = 2rp$, $h_2 = h_5 = h_6 = 0$, $h_3 = 2pq$, $h_4 = q^2$ and $p^2 = -2rq$, then $\phi(\xi)$ is one of the 12 $\phi_l^{\mathrm{II}}(l = 1, 2, \ldots, 12)$

For example, if we select $l = 12$, from case 1 we obtain soliton-like solutions of equations (27) and (28):

$$u = -\frac{2k_1k_3qr + k_4 f_1(y)}{k_1k_3} - \frac{1}{4}k_1k_3\varepsilon\sqrt{-2qr}\, f_1(y)\,\mathrm{sech}(N\xi)\,\mathrm{csch}(N\xi)$$

$$\times[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N]$$

$$-\frac{1}{16}k_1k_3 f_1(y)\,\mathrm{sech}^2(N\xi)\,\mathrm{csch}^2(N\xi)[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi)$$

$$+8N\cosh^2(N\xi) - 4N]^2 - \frac{4k_1k_3qr\varepsilon\sqrt{-2qr}\, f_1(y)\cosh(N\xi)\sinh(N\xi)}{\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N}$$

$$-\frac{16k_1k_3q^2r^2 f_1(y)\cosh^2(N\xi)\sinh^2(N\xi)}{[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N]^2} \pm \frac{3k_1k_3|q|r f_1(y)}{[\sqrt{3}\cosh(2N\xi) \mp \sinh(2N\xi)]^2}$$

$$\pm\frac{3k_1k_3\varepsilon f_1(y)\,\mathrm{sech}^2(N\xi)\,\mathrm{csch}^2(N\xi)[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N]^2}{16[\sqrt{3}\cosh(2N\xi) \mp \sinh(2N\xi)]^2},$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{1}{4}k_1^2k_3^2\varepsilon\sqrt{-2qr}\,\mathrm{sech}(N\xi)\,\mathrm{csch}(N\xi)[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi)$$

$$+8N\cosh^2(N\xi) - 4N] - \frac{1}{16}k_1^2k_3^2\,\mathrm{sech}^2(N\xi)\,\mathrm{csch}^2(N\xi)[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi)$$

$$+8N\cosh^2(N\xi) - 4N]^2 - \frac{4k_1^2k_3^2qr\varepsilon\sqrt{-2qr}\cosh(N\xi)\sinh(N\xi)}{\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N}$$

$$-\frac{16k_1^2k_3^2q^2r^2\cosh^2(N\xi)\sinh^2(N\xi)}{[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N]^2} \pm \frac{3k_1^2k_3^2|q|r}{[\sqrt{3}\cosh(2N\xi) \mp \sinh(2N\xi)]^2}$$

$$\pm\frac{3k_1^2k_3^2\varepsilon\,\mathrm{sech}^2(N\xi)\,\mathrm{csch}^2(N\xi)[\mp 2\sqrt{-2qr}\sinh(N\xi)\cosh(N\xi) + 8N\cosh^2(N\xi) - 4N]^2}{16[\sqrt{3}\cosh(2N\xi) \mp \sinh(2N\xi)]^2},$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,\mathrm{d}y + f_2(t)$, $N = \sqrt{-6qr}/4$, $qr < 0$.

3.3. If $h_0 = h_1 = h_5 = h_6 = 0$, h_2, h_3, h_4 are arbitrary constants, then $\phi(\xi)$ is one of the ten $\phi_l^{\rm III}(l = 1, 2, \ldots, 10)$

For example, if we select $l = 4$, then $h_2 = 4$, $h_3 = 4(d - 2b)/a$, $h_4 = (c^2 + 4b^2 - 4bd)/a^2$, from case 1 we obtain soliton-like solutions of equations (27) and (28):

$$u = -\frac{(4k_1^2k_3^2 + 3k_4) f_1(y)}{3k_1k_3} - \frac{2k_1k_3(d - 2b) f_1(y) \operatorname{csch}^2(\xi)}{b \operatorname{csch}^2(\xi) + c \coth(\xi) + d} - \frac{k_1k_3(c^2 + 4b^2 - 4bd) f_1(y) \operatorname{csch}^4(\xi)}{[b \operatorname{csch}^2(\xi) + c \coth(\xi) + d]^2} \mp \frac{4k_1k_3\varepsilon\sqrt{c^2 + 4b^2 - 4bd}\, f_1(y)[c \cosh(2\xi) + d \sinh(2\xi)]}{[2b - d + d \cosh(2\xi) + c \sinh(2\xi)]^2},$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{2k_1^2k_3^2(d - 2b) \operatorname{csch}^2(\xi)}{b \operatorname{csch}^2(\xi) + c \coth(\xi) + d} - \frac{k_1^2k_3^2(c^2 + 4b^2 - 4bd) \operatorname{csch}^4(\xi)}{[b \operatorname{csch}^2(\xi) + c \coth(\xi) + d]^2} \mp \frac{4k_1^2k_3^2\varepsilon\sqrt{c^2 + 4b^2 - 4bd}[c \cosh(2\xi) + d \sinh(2\xi)]}{[2b - d + d \cosh(2\xi) + c \sinh(2\xi)]^2},$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\, \mathrm{d}y + f_2(t)$.

3.4. If $h_1 = h_3 = h_5 = h_6 = 0$, h_0, h_2, h_4 are arbitrary constants, then $\phi(\xi)$ is one of the 16 $\phi_l^{\rm IV}(l = 1, 2, \ldots, 16)$

For example, if we select $l = 13$, then $h_0 = 1/4$, $h_2 = (1 - 2m^2)/2$, $h_4 = 1/4$, from case 1 we obtain combined non-degenerative Jacobi elliptic doubly-like periodic solutions of equations (27) and (28):

$$u = \frac{\left[-k_1^2k_3^2(1 - 2m^2) \pm 3k_1^2k_3^2 - 6k_4\right] f_1(y)}{6k_1k_3} - \frac{1}{4}\frac{k_1k_3 f_1(y)}{[\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)]^2} - \frac{1}{4}k_1k_3 f_1(y)[\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)]^2 \mp \frac{1}{2}k_1k_3 f_1(y)[\mathrm{cs}(\xi)\,\mathrm{ds}(\xi) \pm \mathrm{ns}(\xi)\,\mathrm{ds}(\xi)] \pm \frac{1}{2}k_1k_3 f_1(y)\frac{\mp\mathrm{ds}(\xi)}{\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)},$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{1}{4}\frac{k_1^2k_3^2}{[\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)]^2} - \frac{1}{4}k_1^2k_3^2[\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)]^2 \mp \frac{1}{2}k_1^2k_3^2[\mathrm{cs}(\xi)\,\mathrm{ds}(\xi) \pm \mathrm{ns}(\xi)\,\mathrm{ds}(\xi)] \pm \frac{1}{2}k_1^2k_3^2\frac{\mp\mathrm{ds}(\xi)}{\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)},$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\, \mathrm{d}y + f_2(t)$.

In the limit case when $m \to 1$, we obtain combined soliton-like solutions of equations (27) and (28):

$$u = \frac{\left(k_1^2k_3^2 \pm 3k_1^2k_3^2 - 6k_4\right) f_1(y)}{6k_1k_3} - \frac{1}{4}\frac{k_1k_3 f_1(y)}{[\coth(\xi) \pm \operatorname{csch}(\xi)]^2} - \frac{1}{4}k_1k_3 f_1(y)[\coth(\xi) \pm \operatorname{csch}(\xi)]^2 \mp \frac{1}{2}k_1k_3 f_1(y)[\operatorname{csch}^2(\xi) \pm \coth(\xi) \operatorname{csch}(\xi)] \pm \frac{1}{2}k_1k_3 f_1(y)\frac{\mp\operatorname{csch}(\xi)}{\coth(\xi) \pm \operatorname{csch}(\xi)},$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{1}{4}\frac{k_1^2k_3^2}{[\coth(\xi) \pm \operatorname{csch}(\xi)]^2} - \frac{1}{4}k_1^2k_3^2[\coth(\xi) \pm \operatorname{csch}(\xi)]^2 \mp \frac{1}{2}k_1^2k_3^2[\operatorname{csch}^2(\xi) \pm \coth(\xi) \operatorname{csch}(\xi)] \pm \frac{1}{2}k_1^2k_3^2\frac{\mp\operatorname{csch}(\xi)}{\coth(\xi) \pm \operatorname{csch}(\xi)},$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\, \mathrm{d}y + f_2(t)$.

When $m \to 0$, we obtain triangular-like solutions of equations (27) and (28):

$$u = \frac{\left(-k_1^2k_3^2 \pm 3k_1^2k_3^2 - 6k_4\right)f_1(y)}{6k_1k_3} - \frac{1}{4}\frac{k_1k_3f_1(y)}{[\csc(\xi) \pm \cot(\xi)]^2} - \frac{1}{4}k_1k_3f_1(y)[\csc(\xi) \pm \cot(\xi)]^2$$
$$\mp \frac{1}{2}k_1k_3f_1(y)[\cot(\xi)\csc(\xi) \pm \csc^2(\xi)] \pm \frac{1}{2}k_1k_3f_1(y)\frac{\mp\csc(\xi)}{\csc(\xi) \pm \cot(\xi)},$$
$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{1}{4}\frac{k_1^2k_3^2}{[\csc(\xi) \pm \cot(\xi)]^2} - \frac{1}{4}k_1^2k_3^2[\csc(\xi) \pm \cot(\xi)]^2$$
$$\mp \frac{1}{2}k_1^2k_3^2[\cot(\xi)\csc(\xi) \pm \csc^2(\xi)] \pm \frac{1}{2}k_1^2k_3^2\frac{\mp\csc(\xi)}{\csc(\xi) \pm \cot(\xi)},$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,dy + f_2(t)$.

3.5. If $h_2 = h_4 = h_5 = h_6 = 0$, h_0, h_1, h_3 are arbitrary constants, then $\phi(\xi)$ is the only ϕ_1^{V}

From equation (37) we get $h_0 = 0$ or $h_3 = 0$, equations (27) and (28) have not solutions for this case. Fortunately, from case 2 we obtain Weierstrass elliptic doubly-like periodic solutions of equations (27) and (28):

$$u = -\frac{k_4}{k_1k_3}f_1(y) - \frac{1}{2}k_1k_3h_3f_1(y)\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right),$$
$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{1}{2}k_1^2k_3^2h_3\wp\left(\frac{\sqrt{h_3}}{2}\xi, g_2, g_3\right),$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,dy + f_2(t)$, $h_3 > 0$, $g_2 = -4h_1/h_3$, $g_3 = -4h_0/h_3$.

From (29) and (30), cases 3–4 and cases I–V listed in the present paper, we can obtain many kinds of solutions of equations (27) and (28) depending on the special choice for h_i $(i = 0, 1, 2, \ldots, 6)$.

3.6. If $h_1 = h_3 = h_5 = 0$, $h_0 = \frac{8h_2^2}{27h_4}$ and $h_6 = \frac{h_4^2}{4h_2}$, then $\phi(\xi)$ is one of the (9) and (10)

For example, if we select (9), from case 3 we obtain triangular-like solutions (see figures 1 and 2) of equations (27) and (28):

$$u = -\frac{\left(4k_1^2k_3^2h_2 + 3k_4\right)f_1(y)}{3k_1k_3} - \frac{16k_1k_3h_2f_1(y)\tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{3[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]}$$
$$- \frac{64k_1k_3h_2f_1(y)\tan^4(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{9[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]^2} \mp \frac{8k_1k_3h_2\varepsilon f_1(y)\sin(2\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{\sqrt{3}[1 + 2\cos(2\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]^2}, \quad (56)$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{16k_1^2k_3^2h_2\tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{3[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]}$$
$$- \frac{64k_1^2k_3^2h_2\tan^4(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{9[3 - \tan^2(\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]^2} \mp \frac{8k_1^2k_3^2h_2\varepsilon\sin(2\varepsilon\sqrt{h_2/3}(\xi+\xi_0))}{\sqrt{3}[1 + 2\cos(2\varepsilon\sqrt{h_2/3}(\xi+\xi_0))]^2}, \quad (57)$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,dy + f_2(t)$.

3.7. If $h_0 = h_1 = h_3 = h_5 = 0$ and $h_6 \neq 0$, then $\phi(\xi)$ is one of the (11)–(19) and (24)

For example, if we select (12), from case 3 we obtain soliton-like solutions of equations (27) and (28):

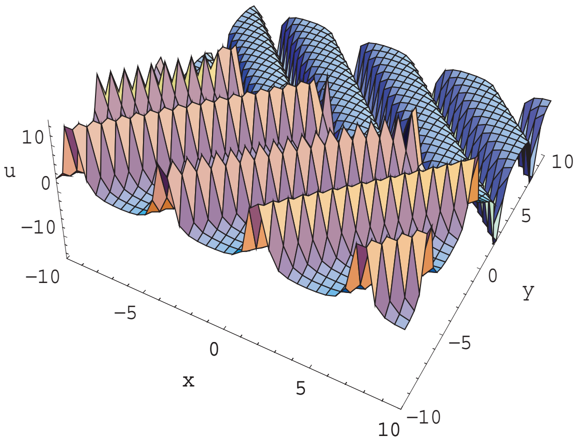

Figure 1. Spatial structure of equation (56) is shown at $k_1 = k_2 = k_3 = k_4 = \xi_0 = \varepsilon = 1, h_2 = 1$, $f_1(y) = \tanh(y)$, $f_2(t) = \mathrm{sech}(t)$, $t = 0$, and the sign '$\mp$' selected by '+'.

$$u = -\frac{\left(4k_1^2k_3^2h_2 + 3k_4\right) f_1(y)}{3k_1k_3} - \frac{2k_1k_3h_2h_4^2 f_1(y)\,\mathrm{csch}^2(\sqrt{h_2}(\xi+\xi_0))}{h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2}$$
$$- \frac{4k_1k_3h_2^2h_4^2h_6 f_1(y)\,\mathrm{csch}^4(\sqrt{h_2}(\xi+\xi_0))}{\left[h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2\right]^2} \pm 2k_1k_3h_2h_4\sqrt{h_2h_6} f_1(y)\,\mathrm{csch}^4(\sqrt{h_2}(\xi+\xi_0))$$
$$\times \frac{[2h_2h_6\varepsilon\cosh(2\sqrt{h_2}(\xi+\xi_0)) + (2h_2h_6 - h_4^2)\sinh(2\sqrt{h_2}(\xi+\xi_0))]}{\left[h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2\right]^2},$$

$$v = \frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} - \frac{2k_1^2k_3^2h_2h_4^2\,\mathrm{csch}^2(\sqrt{h_2}(\xi+\xi_0))}{h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2}$$
$$- \frac{4k_1^2k_3^2h_2^2h_4^2h_6\,\mathrm{csch}^4(\sqrt{h_2}(\xi+\xi_0))}{\left[h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2\right]^2} \pm 2k_1^2k_3^2h_2h_4\sqrt{h_2h_6}\,\mathrm{csch}^4(\sqrt{h_2}(\xi+\xi_0))$$
$$\times \frac{[2h_2h_6\varepsilon\cosh(2\sqrt{h_2}(\xi+\xi_0)) + (2h_2h_6 - h_4^2)\sinh(2\sqrt{h_2}(\xi+\xi_0))]}{[h_4^2 - h_2h_6(1+\varepsilon\coth(\sqrt{h_2}(\xi+\xi_0)))^2]^2},$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

3.8. If $h_0 = h_1 = h_3 = h_5 = 0$, $h_6 \neq 0$ and $h_4^2 - 4h_2h_6 = 0$, then $\phi(\xi)$ is one of the (25) and (26)

For example, if we select (25), from case 3 we obtain soliton-like solutions of equations (27) and (28):

$$u = -\frac{\left(4k_1^2k_3^2h_2 + 3k_4\right) f_1(y)}{3k_1k_3} + 2k_1k_3h_2 f_1(y)[1+\varepsilon\tanh(\varepsilon\sqrt{h_2}(\xi+\xi_0))]$$
$$- k_1k_3h_2 f_1(y)[1+\varepsilon\tanh(\varepsilon\sqrt{h_2}(\xi+\xi_0))]^2 \mp k_1k_3h_2\varepsilon f_1(y)\,\mathrm{sech}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0)),$$
$$v = -\frac{3k_1k_3k_4 + f_2'(t)}{3k_1k_3} + 2k_1^2k_3^2h_2[1+\varepsilon\tanh(\varepsilon\sqrt{h_2}(\xi+\xi_0))]$$
$$- k_1^2k_3^2h_2[1+\varepsilon\tanh(\varepsilon\sqrt{h_2}(\xi+\xi_0))]^2 \mp k_1^2k_3^2h_2\varepsilon\,\mathrm{sech}^2(\varepsilon\sqrt{h_2}(\xi+\xi_0)),$$

where $\xi = (k_1x + k_2)k_3 + \int f_1(y)\,\mathrm{d}y + f_2(t)$.

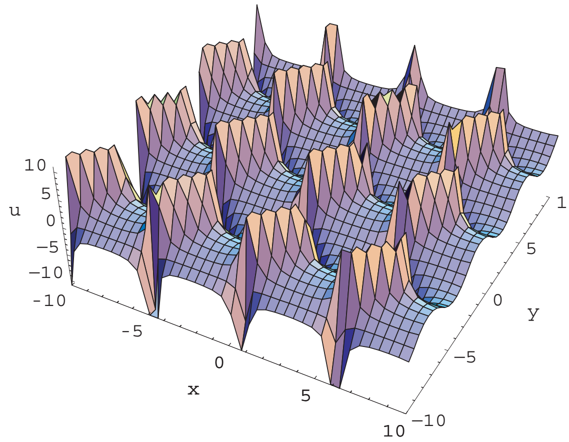

Figure 2. Spatial structure of equation (56) is shown at $k_1 = k_2 = k_3 = k_4 = \xi_0 = \varepsilon = 1$, $h_2 = 1$, $f_1(y) = \sin(y)$, $f_2(t) = \tanh(t)$, $t = 0$, and the sign '$\mp$' selected by '+'.

From (29) and (30), cases 1–4, we can obtain other exact solutions of equations (27) and (28), here we omit them for simplicity.

Remark 1. Chen *et al* obtained only case 2 in [21]. To the best of our knowledge, all the solutions obtained from cases 1, 3 and 4 are new and have not been reported yet. All the results reported in this paper have been checked with *Mathematica*. By using our method, we can also obtain new and more general exact solutions of the other NLPDEs in [20, 22–29] including all the solutions given there as special cases of our method. It shows that our method is more powerful than the methods [20–29] in constructing exact solutions of NLPDEs.

4. Conclusion

In this paper, we have presented a generalized auxiliary equation method to construct more general exact solutions of NLPDEs, which can be thought of as the expansion of tanh function method [6], F-expansion method [16, 17], algebraic method [20–23], auxiliary equation method [24–29]. With the help of *Mathematica*, our method provides a powerful mathematical tool to obtain more general exact solutions of a great many NLPDEs in mathematical physics, such as the (3+1)-dimensional Kadomtsev–Petviashvili equation, the (2+1)-dimensional Korteweg–de Vries equations, Broer–Kaup–Kupershmidt equations, breaking soliton equations, Broer–Kaup equations, dispersive long wave equations and so on. Applying our method to the (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations, we have obtained many new and more general exact solutions with two arbitrary functions. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. It may be important to explain some physical phenomena.

It should be noted that more complicated computation is expected than ever before because of using the general ansatz (2). In general it is very difficult to solve the set of over-determined partial differential equations obtained in step 3. As the calculation goes on, in order to drastically simplify the work or make the work feasible, we often choose special forms for a_0, a_i, b_i, c_i, d_i and ξ on a trial and error basis. In appendix A, the KdV equation (4) is considered. Besides, for some special types of NLPDEs, such as nonlinear Schrödinger equation, sine-Gordon equation, Tzitzeica–Dodd–Bullough equation and so on, we can take some proper transformations to change them into convenient ones for us to use our method. In appendix B, three examples are given.

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Appendix A

For the KdV equation (4), we assume the solution of it can be expressed by

$$u = a_0 + \sum_{i=1}^{4} \{a_i \phi^{-i}(\xi) + b_i \phi^{i}(\xi) + c_i \phi^{i-1}(\xi)\phi'(\xi) + d_i \phi^{-i}(\xi)\phi'(\xi)\}, \tag{A.1}$$

where $a_0 = a_0(x,t)$, $a_i = a_i(x,t)$, $b_i = b_i(x,t)$, $c_i = c_i(x,t)$, $d_i = d_i(x,t) (i = 1, 2, 3, 4)$, $\xi = \xi(x,t)$.

With the aid of *Mathematica*, substituting (A.1) along with (3) into equation (4), then collecting the coefficients of $\phi^j(\xi)\phi'^l(\xi)(l = 0, 1; j = \pm 1, \pm 2, \ldots)$ to zero, we get a set of over-determined partial differential equations for $a_0, a_i, b_i, c_i, d_i, \xi$ as follows:

$$\begin{aligned}
&33c_4\xi_x(2h_6c_3 + h_5c_4) = 0,\\
&6h_6c_4\left(10b_4\xi_x + 80h_6\xi_x^3 + c_{4,x}\right) = 0, \qquad -48h_0a_4d_4\xi_x = 0,\\
&15\xi_x\left[-h_1d_3^2 - h_3d_4^2 - 6h_1a_4\xi_x^2 - 2(a_1a_4 + a_2a_3 + h_0d_2d_3 + h_0d_1d_4 + h_1d_2d_4 + h_2d_3d_4)\right.\\
&\qquad\left. + 4h_0\left(d_4\xi_{xx} + d_{4,x}\xi_x + a_3\xi_x^2\right)\right] + 6(a_2d_4 + a_3d_3 + a_4d_2)_x = 0,\\
&d_{1,t} + d_{1,xxx} + 6(a_0d_1 + a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4 + b_1d_2 + b_2d_3 + b_3d_4)_x = 0,\\
&24\xi_x\left[b_4^2 + h_6c_2^2 + h_4c_3^2 + h_2c_4^2 + 2(h_6c_1c_3 + h_5c_1c_4 + h_5c_2c_3 + h_4c_2c_4 + h_3c_3c_4 + h_6c_4d_1)\right.\\
&\qquad\left. + 2h_6\xi_x\left(3c_4\xi_{xx} + 3c_{4,x}\xi_x + 4b_4\xi_x^2\right)\right] + 6(b_4c_4)_x = 0,\\
&-24\xi_x\left(a_4^2 + h_0d_4^2\right) = 0, \qquad 36h_6c_4^2\xi_x = 0,\\
&\vdots
\end{aligned}$$

there are totally 43 equations in the set of over-determined partial differential equations, just some simple and central equations are shown here for convenience. However, it is very difficult for us to get the explicit expressions for a_0, a_i, b_i, c_i, d_i and ξ from the set of over-determined partial differential equations. For example, one result is obtained as follows:

$$a_0 = \frac{-4h_2\xi_x^4 - \xi_x\xi_t + 3\xi_{xx}^2 - 4\xi_x\xi_{xxx}}{6\xi_x^2}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0,$$
$$a_4 = 0, \qquad b_1 = 0, \tag{A.2}$$

$$b_2 = -2h_4\xi_x^2 \pm 2\sqrt{h_6}\xi_{xx}, \qquad b_3 = 0, \qquad b_4 = -4h_6\xi_x^2, \qquad c_1 = 0,$$
$$c_2 = \pm 4\sqrt{h_6}\xi_x^2, \qquad c_3 = 0, \tag{A.3}$$

$$c_4 = 0, \qquad d_1 = -2\xi_{xx}, \qquad d_2 = 0, \qquad d_3 = 0, \qquad d_4 = 0, \qquad h_0 = 0,$$
$$h_1 = 0, \qquad h_3 = 0, \qquad h_5 = 0, \tag{A.4}$$

where ξ satisfies

$$\xi_x\xi_{xx}(\xi_t+4\xi_{xxx})-\xi_x^2(\xi_{xt}+\xi_{xxxx})+4h_2\xi_x^4\xi_{xx}-3\xi_{xx}^3=0, \tag{A.5}$$

$$\begin{aligned}&15\xi_x\xi_{xx}^3(\xi_t+16\xi_{xxx})-\xi_x^2\xi_{xx}\left(\xi_t^2+15\xi_{xt}\xi_{xx}+20\xi_t\xi_{xxx}+78\xi_{xx}\xi_{xxxx}+136\xi_{xxx}^2\right)\\&\quad+\xi_x^3(2\xi_t\xi_{xt}+9\xi_{xx}\xi_{xxt}+11\xi_{xt}\xi_{xxx}+5\xi_t\xi_{xxxx}+50\xi_{xxx}\xi_{xxxx}+18\xi_{xx}\xi_{xxxxx})\\&\quad+32h_2^2\xi_x^8\xi_{xx}-\xi_x^4(\xi_{tt}+5\xi_{xxxt}+4\xi_{xxxxxx})+4h_2\xi_x^5\xi_{xx}(\xi_t+28\xi_{xxx})\\&\quad-4h_2\xi_x^6(\xi_{xt}-2\xi_{xxxx})-90\xi_{xx}^5=0,\end{aligned} \tag{A.6}$$

$$\begin{aligned}&\sqrt{h_0}\xi_x\xi_{xx}^2(\xi_t+13\xi_{xxx})-\sqrt{h_6}\xi_x^2\left(\xi_{xt}\xi_{xx}+3h_4\xi_{xx}^3+\xi_t\xi_{xxx}+4\xi_{xxx}^2+4\xi_{xx}\xi_{xxx}\right)\\&\quad+\xi_x^3(h_4\xi_t\xi_{xx}+\sqrt{h_6}\xi_{xxt}+4h_4\xi_{xx}\xi_{xxx}+\sqrt{h_6}\xi_{xxxxx})\\&\quad-\xi_x^4(h_4\xi_{xt}+8h_2\sqrt{h_6}\xi_{xx}^2+h_4\xi_{xxxx})-4h_2\sqrt{h_6}\xi_x^5\xi_{xxx}\\&\quad+4h_2h_4\xi_x^6\xi_{xx}-6\sqrt{h_6}\xi_{xx}^4=0,\end{aligned} \tag{A.7}$$

$$\begin{aligned}&-\xi_x\xi_{xx}^2(\xi_t+13\xi_x\xi_{xxx})+\xi_x^2\left(\xi_{xt}\xi_{xx}+\xi_t\xi_{xxx}+4\xi_{xxx}^2+4\xi_{xx}\xi_{xxxx}\right)+8h_2\xi_x^4\xi_{xx}^2\\&\quad-\xi_x^3(\xi_{xxt}+\xi_{xxxxxx})+4h_2\xi_x^5\xi_{xxx}+6\xi_{xx}^4=0.\end{aligned} \tag{A.8}$$

But it is not easy for us to get the explicit expression for ξ from equations (A.5)–(A.8). In order to make the work feasible, we further set

$$\xi=p+q,\qquad p=p(x),\qquad q=q(t), \tag{A.9}$$

then equations (A.5)–(A.8) are equivalent to the following equation:

$$p^{(4)}p'^2+p''(-4h_2p'^4-p'q'+3p''^2-4p'p^{(3)})=0,\qquad q''=0. \tag{A.10}$$

It is obvious that equation (A.10) has one solution by introducing the constants k, ω, k_1 and k_2

$$p=kx+k_1,\qquad q=\omega t+k_2, \tag{A.11}$$

from which a_0, b_2, b_4, c_2 and d_1 can be determined exactly.

Appendix B

If the F given in equation (1) is not a polynomial in real number field, we can use exponential function to change equation (1) into two polynomials in real number field by separating the real and imaginary parts. If the F is not a polynomial of u and its partial derivatives, we can take a proper transformation by introducing a new variable, for example, v to change equation (1) into a polynomial of v and its partial derivatives. We next give three examples to illustrate the effectiveness of our method in solving some special types of NLPDEs as mentioned here.

First, let us consider the variable coefficient nonlinear Schrödinger equation [30], which reads

$$\mathrm{i}\psi_z+\tfrac{1}{2}\alpha(z)\psi_{tt}+\beta(z)|\psi|^2\psi=\mathrm{i}\gamma(z)\psi, \tag{B.1}$$

where $\psi=\psi(z,t)$ is a real or complex-valued arbitrary function of z and t, $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ are all arbitrary functions of indicated variable. Equation (B.1) is the nonlinear Schrö dinger equation with gain in the form used in nonlinear fibre optics. In order to obtain exact solution of equation (B.1), we make the transformation

$$\psi(z,t)=A(z,t)\exp[\mathrm{i}\theta(z,t)], \tag{B.2}$$

where $A(z,t)$ and $\theta(z,t)$ are amplitude and phase functions, respectively. Substituting (B.2) into equation (B.1) and separating the real and imaginary parts, we obtain

$$-A\theta_z + \tfrac{1}{2}\alpha(z)\left(A_{tt} - A\theta_t^2\right) + \beta(z)A^3 = 0, \tag{B.3}$$

$$A_z + \tfrac{1}{2}\alpha(z)(2A_t\theta_t + A\theta_{tt}) - \gamma(z)A = 0. \tag{B.4}$$

Balancing A_{tt} and A^3 in equation (B.3), we have $n = 2$. We assume that equations (B.3) and (B.4) have the formal solution expressed by

$$A = a_0 + \sum_{i=1}^{2}\{a_i\phi^{-i}(\xi) + b_i\phi^i(\xi) + c_i\phi^{i-1}(\xi)\phi'(\xi) + d_i\phi^{-i}(\xi)\phi'(\xi)\}, \tag{B.5}$$

where $a_0 = a_0(z,t), a_i = a_i(z,t), b_i = b_i(z,t), c_i = c_i(z,t), d_i = d_i(z,t) (i = 1, 2), \xi = p + q, p = p(z), q = q(t)$.

Substituting (B.5) along with (3) into equations (B.3) and (B.4), then collecting the coefficients of $\phi^j(\xi)\phi'^l(\xi)$ $(l = 0, 1;\ j = \pm 1, \pm 2, \ldots)$ to zero, we get a set of over-determined partial differential equations for $a_0, a_i, b_i, c_i, d_i, p, q$ and θ as follows:

$$c_2^2\beta(z)(3h_6c_1 + h_5c_2) = 0,$$
$$d_2^2\beta(z)\left(3a_2^2 + h_0d_2^2\right) = 0, \qquad h_6c_2^3\beta(z) = 0,$$
$$c_{1,z} - c_1\gamma(z) + b_1p' + b_1\alpha(z)q'\theta_t + \alpha(z)c_{1,t}\theta_t + \tfrac{1}{2}c_1\alpha(z)\theta_{tt} = 0,$$
$$b_{2,z} - b_2\gamma(z) + b_{2,t}\alpha(z)\theta_t + \tfrac{1}{2}b_2\alpha(z)\theta_{tt} + \tfrac{1}{2}[p' + \alpha(z)q'\theta_t](3h_3c_1 + 4h_2c_2 + 2h_4d_1 + h_5d_2) = 0,$$
$$a_{0,z} - a_0\gamma(z) + a_{0,,t}\alpha(z)\theta_t + \tfrac{1}{2}a_0\alpha(z)\theta_{tt} + \tfrac{1}{2}[p' + \alpha(z)q'\theta_t](h_1c_1 - h_3d_2 + 2h_0c_2) = 0,$$
$$d_{2,z} - d_2\gamma(z) - a_1p' - a_1\alpha(z)q'\theta_t + \alpha(z)d_{2,t}\theta_t + \tfrac{1}{2}d_2\alpha(z)\theta_{tt} = 0,$$
$$d_{1,z} - d_1\gamma(z) + d_{1,t}\alpha(z)\theta_t + \tfrac{1}{2}d_1\alpha(z)\theta_{tt} = 0,$$
$$a_2\beta(z)\left(a_2^2 + 3h_0d_2^2\right) = 0,$$
$$\vdots$$

there are totally 48 equations in the set of over-determined partial differential equations, just some simple and central equations are shown here for convenience. Solving the system of over-determined partial differential equations by use of *Mathematica*, we obtain the following results.

Case 1.1.

$$a_0 = 0, \qquad a_1 = \pm\frac{\omega}{2}\sqrt{-\frac{h_0\alpha(z)}{\beta(z)}}, \qquad a_2 = 0, \qquad b_1 = \pm\frac{\omega}{2}\sqrt{-\frac{h_4\alpha(z)}{\beta(z)}}, \qquad b_2 = 0, \tag{B.6}$$

$$c_1 = 0, \qquad c_2 = 0, \qquad d_1 = \pm\frac{\omega}{2}\sqrt{-\frac{\alpha(z)}{\beta(z)}}, \qquad d_2 = 0, \qquad h_5 = 0, \qquad h_6 = 0, \tag{B.7}$$

$$p = \delta\omega^2\sqrt{-\frac{h_2}{2} \pm 3\sqrt{h_0h_4}}\int \alpha(z)\,dz, \qquad \theta = -\delta\omega\sqrt{-\frac{h_2}{2} \pm 3\sqrt{h_0h_4}}\,t + k_1, \tag{B.8}$$

$$\gamma(z) = \frac{\beta(z)\alpha'(z) - \alpha(z)\beta'(z)}{2\alpha(z)\beta(z)}, \qquad \pm h_3\sqrt{h_0} - h_1\sqrt{h_4} = 0, \qquad q = \omega t + k_2, \tag{B.9}$$

where $\alpha'(z) = \mathrm{d}\alpha(z)/\mathrm{d}z$, $\beta'(z) = \mathrm{d}\beta(z)/\mathrm{d}z$, $\delta = \pm 1$, ω, k_1 and k_2 are arbitrary constants. The sign '$\pm$' in a_1, b_1 and d_1 means that all possible combinations of '+' and '$-$' can be taken. If the same sign is used in a_1, b_1, and $\omega > 0$, then '$-$' must be used in p, θ and (B.9). If the same sign is used in a_1, b_1, and $\omega < 0$, then '+' must be used in p, θ and (B.9). If different signs are used in a_1, b_1, and $\omega > 0$, then '+' must be used in θ, p and (B.9). If different signs are used in a_1, b_1, and $\omega < 0$, then '$-$' must be used in p, θ and (B.9).

Case 1.2.

$$a_0 = \pm\frac{h_3 w}{4h_4}\sqrt{-\frac{h_4\alpha(z)}{\beta(z)}}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad b_1 = \pm\omega\sqrt{-\frac{h_4\alpha(z)}{\beta(z)}},$$
$$b_2 = 0, \qquad c_1 = 0, \tag{B.10}$$

$$c_2 = 0, \qquad d_1 = 0, \qquad d_2 = 0, \qquad h_5 = 0, \qquad h_6 = 0,$$
$$p = \delta\omega^2\sqrt{h_2 - \frac{3h_3^2}{8h_4}}\int \alpha(z)\,\mathrm{d}z, \qquad q = \omega t + k_2, \tag{B.11}$$
$$\gamma(z) = \frac{\beta(z)\alpha'(z) - \alpha(z)\beta'(z)}{2\alpha(z)\beta(z)}, \qquad h_3^3 - 4h_2h_3h_4 + 8h_1h_4^2 = 0,$$
$$\theta = -\delta\omega\sqrt{h_2 - \frac{3h_3^2}{8h_4}}\,t + k_1, \tag{B.12}$$

where $\alpha'(z) = \mathrm{d}\alpha(z)/\mathrm{d}z$, $\beta'(z) = \mathrm{d}\beta(z)/\mathrm{d}z$, $\delta = \pm 1$, ω, k_1 and k_2 are arbitrary constants.

Case 1.3.

$$a_0 = 0, \qquad a_1 = 0, \qquad a_2 = 0, \qquad b_1 = 0, \qquad b_2 = \pm\omega\sqrt{-\frac{h_6\alpha(z)}{\beta(z)}},$$
$$c_1 = 0, \qquad c_2 = 0, \tag{B.13}$$

$$d_1 = \pm\omega\sqrt{-\frac{\alpha(z)}{\beta(z)}}, \qquad d_2 = 0, \qquad h_0 = 0, \qquad h_1 = 0, \qquad h_3 = 0, \qquad h_5 = 0,$$
$$p = \delta\omega^2\sqrt{-2h_2}\int \alpha(z)\,\mathrm{d}z, \tag{B.14}$$
$$\gamma(z) = \frac{\beta(z)\alpha'(z) - \alpha(z)\beta'(z)}{2\alpha(z)\beta(z)}, \qquad \theta = -\delta\omega\sqrt{-2h_2}\,t + k_1, \qquad q = \omega t + k_2, \tag{B.15}$$

where $\alpha'(z) = \mathrm{d}\alpha(z)/\mathrm{d}z$, $\beta'(z) = \mathrm{d}\beta(z)/\mathrm{d}z$, $\delta = \pm 1$, ω, k_1 and k_2 are arbitrary constants.

Case 1.4.

$$a_0 = \pm\frac{h_4\omega}{2h_6}\sqrt{-\frac{h_6\alpha(z)}{\beta(z)}}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad b_1 = 0,$$
$$b_2 = \pm 2\omega\sqrt{-\frac{h_6\alpha(z)}{\beta(z)}}, \qquad c_1 = 0, \qquad c_2 = 0, \tag{B.16}$$

$$d_1 = 0, \qquad d_2 = 0, \qquad h_1 = 0, \qquad h_3 = 0, \qquad h_5 = 0,$$
$$p = \delta\omega^2\sqrt{4h_2 - \frac{3h_4^2}{2h_6}}\int \alpha(z)\,\mathrm{d}z, \qquad q = \omega t + k_2, \tag{B.17}$$

$$\gamma(z)=\frac{\beta(z)\alpha'(z)-\alpha(z)\beta'(z)}{2\alpha(z)\beta(z)},\qquad h_4^3-4h_2h_4h_6+8h_0h_6^2=0,$$
$$\theta=-\delta\omega\sqrt{4h_2-\frac{3h_4^2}{2h_6}}t+k_1,\tag{B.18}$$

where $\alpha'(z)=\mathrm{d}\alpha(z)/\mathrm{d}z$, $\beta'(z)=\mathrm{d}\beta(z)/\mathrm{d}z$, $\delta=\pm1$, ω, k_1 and k_2 are arbitrary constants.

From (15), (B.2), (B.5) and case 1.3, we obtain exact solution of equation (B.1):

$$\psi(z,t)=\omega\sqrt{-\frac{\alpha(z)}{\beta(z)}}\left[\frac{(\mp1\mp\varepsilon)h_2\sqrt{h_6}\sec^2(\xi+\xi_0)}{h_4+2\varepsilon\sqrt{-h_2h_6}\tan(\xi+\xi_0)}\mp\sqrt{-h_2}\tan(\xi+\xi_0)\right]$$
$$\times\exp[\mathrm{i}(-\delta\omega\sqrt{-2h_2}t+k_1)],$$

where $\xi=\delta\omega^2\sqrt{-2h_2}\int\alpha(z)\,\mathrm{d}z+\omega t+k_2$.

Second, we consider the Tzitzeica–Dodd–Bullough equation [31]:

$$u_{xt}=\mathrm{e}^u+\mathrm{e}^{-2u},\tag{B.19}$$

which plays a significant role in many scientific applications such as solid-state physics, nonlinear optics and quantum field theory. By making the transformation

$$v(x,t)=\mathrm{e}^{-u},\qquad u(x,t)=\operatorname{arcsinh}\left[\frac{v^{-1}-v}{2}\right],\tag{B.20}$$

equation (B.19) becomes

$$-vv_{xt}+v_xv_t-v^3-v^4=0.\tag{B.21}$$

Balancing vv_{xt} and v^4 in equation (B.21), we have $n=2$. We assume equation (B.21) has solution in the form:

$$v=a_0+\sum_{i=1}^{2}\{a_i\phi^{-i}(\xi)+b_i\phi^i(\xi)+c_i\phi^{i-1}(\xi)\phi'(\xi)+d_i\phi^{-i}(\xi)\phi'(\xi)\},\tag{B.22}$$

where $a_0=a_0(t)$, $a_i=a_i(t)$, $b_i=b_i(t)$, $c_i=c_i(t)$, $d_i=d_i(t)(i=1,2)$, $\xi=\xi(x,t)$.

Substituting equation (B.22) along with equation (3) into equation (B.21), then collecting the coefficients of $\phi^j(\xi)\phi'^l(\xi)(l=0,1;\ j=\pm1,\pm2,\ldots)$ to zero, we get a set of over-determined partial differential equations for a_0, a_i, b_i, c_i, d_i and ξ. There are totally 46 equations in the set of over-determined partial differential equations, we omit them here for convenience. Solving the system of over-determined partial differential equations by use of *Mathematica*, we obtain the following results.

Case 2.1.

$$a_0=-\frac{1}{2},\qquad a_1=0,\qquad a_2=0,\qquad b_1=\pm\frac{1}{2}\sqrt{\frac{h_4}{h_2}},\qquad b_2=0,$$
$$c_1=0,\qquad c_2=0,\tag{B.23}$$

$$d_1=\pm\frac{1}{2}\sqrt{\frac{1}{h_2}},\qquad d_2=0,\qquad h_0=0,\qquad h_1=0,\qquad h_5=0,$$
$$h_6=0,\qquad \xi=kx-\frac{1}{h_2k}t+c,\tag{B.24}$$

where c is an arbitrary constant, k is a nonzero constant. The sign '$\pm$' in b_1 and d_1 means that all possible combinations of '+' and '$-$' can be taken.

Case 2.2.

$$a_0 = -\frac{1}{2}, \qquad a_1 = -\frac{1}{4h_3k\omega}, \qquad a_2 = 0, \qquad b_1 = 0, \qquad b_2 = 0, \qquad c_1 = 0, \tag{B.25}$$

$$c_2 = 0, \qquad d_1 = \pm\frac{1}{2}\sqrt{-k\omega}, \qquad d_2 = 0, \qquad h_0 = -\frac{1}{4h_3^2k^3\omega^3}, \qquad h_1 = -\frac{3}{4h_3k^2\omega^2}, \tag{B.26}$$

$$h_2 = 0, \qquad h_4 = 0, \qquad h_5 = 0, \qquad h_6 = 0, \qquad \xi = kx + \omega t + c, \tag{B.27}$$

where c is an arbitrary constant, k and ω are nonzero constants.

Case 2.3.

$$a_0 = -\frac{1}{2}, \qquad a_1 = 0, \qquad a_2 = 0, \qquad b_1 = 0, \qquad b_2 = \pm\sqrt{-h_6k\omega}, \qquad c_1 = 0, \qquad c_2 = 0, \tag{B.28}$$

$$d_1 = \pm\sqrt{-k\omega}, \qquad d_2 = 0, \qquad h_1 = 0, \qquad h_3 = 0, \qquad h_5 = 0, \qquad h_0 = \pm\frac{(1+4h_2k\omega)\sqrt{-h_6k\omega}}{16h_6k^2\omega^2}, \tag{B.29}$$

$$\left(h_2 + \frac{1}{4k\omega}\right)\left(h_2 - \frac{3h_6 \pm 4h_4\sqrt{-h_6k\omega}}{4h_6k\omega}\right) = 0, \qquad \xi = kx + \omega t + c, \tag{B.30}$$

where c is an arbitrary constant, k and ω are nonzero constants. The sign '$\pm$' in b_2 and d_1 means that all possible combinations of '+' and '$-$' can be taken. If '+' is used in b_2, then '+' must be used in h_0 and (B.30). If '$-$' is used in b_2, then '$-$' must be used in h_0 and (B.30). Hereafter, the sign '$\pm$' always stands for this meaning in the similar circumstances.

If we use case 2.3 with $h_0 \neq 0$ to search for solution of equation (B.19), then from equations (B.29) and (B.30) and the relation of the values of h_0 and h_6 in case I, which reads

$$h_0 = \frac{8h_2^2}{27h_4}, \qquad h_6 = \frac{h_4^2}{4h_2}, \tag{B.31}$$

we obtain

$$h_2 = -\frac{9}{4k\omega}, \tag{B.32}$$

and the condition that if '+' is used in equation (B.30) then $h_4 < 0$, if '$-$' is used in equation (B.30) then $h_4 > 0$.

From equations (B.29) and (B.32), we get $h_2 > 0$ which leads to $h_4 < 0$ if we use case I. Thus, from equations (10), (B.20), (B.22) and (B.32) we obtain exact solution of equation (B.19):

$$u(x,t) = \operatorname{arcsinh}\left[\frac{v^{-1} - v}{2}\right],$$

with

$$v=-\frac{1}{2}-\frac{\cot^2\left(\frac{\varepsilon}{2}\sqrt{-\frac{3}{k\omega}}(\xi+\xi_0)\right)}{3-\cot\left(\frac{\varepsilon}{2}\sqrt{-\frac{3}{k\omega}}(\xi+\xi_0)\right)^2}$$
$$\mp\frac{3\sqrt{3}\varepsilon\sin\left(\varepsilon\sqrt{-\frac{3}{k\omega}}(\xi+\xi_0)\right)\left[3-\cot^2\left(\frac{\varepsilon}{2}\sqrt{-\frac{3}{k\omega}}(\xi+\xi_0)\right)\right]}{4\cot^2\left(\frac{\varepsilon}{2}\sqrt{-\frac{3}{k\omega}}(\xi+\xi_0)\right)\left[1-2\cos\left(\varepsilon\sqrt{-\frac{3}{k\omega}}(\xi+\xi_0)\right)\right]^2},$$

where $\xi=kx+\omega t+c$.

If we use case 2.3 with $h_0=0$ to obtain the solution of equation (B.19), then we get

$$h_2=-\frac{1}{4k\omega}. \tag{B.33}$$

From equations (13), (B.20), (B.22), (B.33) and case 2.3, we obtain exact solution of equation (B.19):

$$u(x,t)=\operatorname{arcsinh}\left[\frac{v^{-1}-v}{2}\right],$$

with

$$v=-\frac{1}{2}+\frac{(\pm1\pm\varepsilon)\sqrt{h_6}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{-\frac{1}{k\omega}}(\xi+\xi_0)\right)}{4\left[h_4\sqrt{-k\omega}-\varepsilon\sqrt{h_6}\tanh\left(\frac{1}{2}\sqrt{-\frac{1}{k\omega}}(\xi+\xi_0)\right)\right]}\mp\frac{1}{2}\tanh\left(\frac{1}{2}\sqrt{-\frac{1}{k\omega}}(\xi+\xi_0)\right),$$

where $\xi=kx+\omega t+c$.

Third, for the sine-Gordon equation [28]:

$$u_{xt}=\sin u, \tag{B.34}$$

which arises classically in the study of differential geometry in mathematics and arises in the study of Josephson junctions, models of particle physics, stability of fluid motions in physics. We make the following transformation

$$v(x,t)=\sin\left[\tfrac{1}{2}u(x,t)\right],\qquad u(x,t)=2\arcsin[v(x,t)], \tag{B.35}$$

then equation (B.34) becomes

$$v^2v_{xt}+v_{xt}-vv_xv_t-v+2v^3-v^5=0. \tag{B.36}$$

Balancing v^2v_{xt} and v^5 in equation (B.36), we have $n=2$. We assume equation (B.36) has solution in the form:

$$v=a_0+\sum_{i=1}^{2}\{a_i\phi^{-i}(\xi)+b_i\phi^i(\xi)+c_i\phi^{i-1}(\xi)\phi'(\xi)+d_i\phi^{-i}(\xi)\phi'(\xi)\}, \tag{B.37}$$

where $a_0=a_0(t), a_i=a_i(t), b_i=b_i(t), c_i=c_i(t), d_i=d_i(t) (i=1,2), \xi=\xi(x,t)$.

By the same manipulation as illustrated above, we obtain the following results.

Case 3.1.

$$a_0=\pm1,\quad a_1=0,\quad a_2=0,\quad b_1=\pm2\sqrt{\frac{h_4}{h_2}},\quad b_2=0,\quad c_1=0,$$
$$c_2=0,\quad d_1=0, \tag{B.38}$$

$$d_2=0,\quad h_0=0,\quad h_1=0,\quad h_5=0,\quad h_6=0,\quad h_3^2-4h_2h_4=0,$$
$$\xi=kx-\frac{4}{h_2k}t+c, \tag{B.39}$$

where k is a nonzero constant, c is an arbitrary constant.

Case 3.2.

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad c_1 = 0, \quad c_2 = 0, \tag{B.40}$$

$$d_1 = \pm\sqrt{-k\omega}, \quad d_2 = 0, \quad h_1 = 0, \quad h_3 = 0, \quad h_5 = 0, \quad h_6 = 0,$$
$$\xi = kx + \omega t + c, \tag{B.41}$$

where c is an arbitrary constant, k and ω are nonzero constants which are determined by

$$1 + 2h_2k\omega + h_2^2k^2\omega^2 - 4h_0h_4k^2\omega^2 = 0. \tag{B.42}$$

Case 3.3.

$$a_0 = \pm 1, \quad a_1 = \pm\frac{4h_0}{h_1}, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad c_1 = 0,$$
$$c_2 = 0, \quad d_1 = 0, \tag{B.43}$$

$$d_2 = 0, \quad h_2 = 0, \quad h_4 = 0, \quad h_5 = 0, \quad h_6 = 0, \quad h_1^3 + 8h_0^2h_3 = 0,$$
$$\xi = kx - \frac{16h_0}{h_1^2k}t + c, \tag{B.44}$$

where c is an arbitrary constant, k is a nonzero constant.

Case 3.4.

$$a_0 = \pm\frac{\sqrt{5}}{5}, \quad a_1 = \pm\frac{4\sqrt{5}h_0}{5h_1}, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = 0, \quad c_1 = 0,$$
$$c_2 = 0, \quad d_1 = 0, \tag{B.45}$$

$$d_2 = 0, \quad h_2 = 0, \quad h_4 = 0, \quad h_5 = 0, \quad h_6 = 0, \quad h_1^3 + 8h_0^2h_3 = 0,$$
$$\xi = kx - \frac{16h_0}{5h_1^2k}t + c, \tag{B.46}$$

where c is an arbitrary constant, k is a nonzero constant.

Case 3.5.

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \pm\sqrt{\frac{h_6}{h_2}},$$
$$c_1 = 0, \quad c_2 = 0, \tag{B.47}$$

$$d_1 = \pm\sqrt{\frac{1}{h_2}}, \quad d_2 = 0, \quad h_0 = 0, \quad h_1 = 0, \quad h_3 = 0, \quad h_5 = 0,$$
$$\xi = kx - \frac{1}{h_2k}t + c, \tag{B.48}$$

where c is an arbitrary constant, k is a nonzero constant.

Case 3.6.

$$a_0 = \pm 1, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \pm 2\sqrt{\frac{h_6}{h_2}}, \quad c_1 = 0,$$
$$c_2 = 0, \quad d_1 = 0, \tag{B.49}$$

$$d_2 = 0, \quad h_0 = 0, \quad h_1 = 0, \quad h_3 = 0, \quad h_5 = 0, \quad h_4^2 - 4h_2h_6 = 0,$$
$$\xi = kx - \frac{1}{h_2 k}t + c, \tag{B.50}$$

where c is an arbitrary constant, k is a nonzero constant.

From equations (14), (B.35), (B.37) and case 3.5, we obtain exact solution of equation (B.34):

$$u(x,t) = 2\arcsin\left\{\frac{(\pm 1 \pm \varepsilon)\sqrt{h_2h_6}\,\mathrm{csch}^2(\sqrt{h_2}(\xi+\xi_0))}{h_2 + 2\varepsilon\sqrt{h_2h_6}\coth(\sqrt{h_2}(\xi+\xi_0))]} \mp \coth(\sqrt{h_2}(\xi+\xi_0))\right\},$$

where $\xi = kx - \frac{1}{h_2k}t + c$.

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